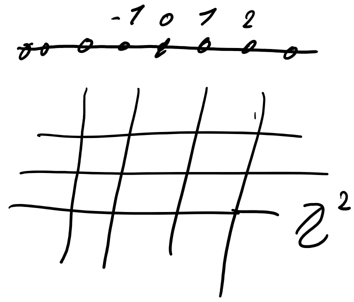
First passage percolation

Thursday, 6 May 2021 13:10



$\mathbb{Z}^d$ – d-dimensional integer lattice

We take $d \geq 2$.

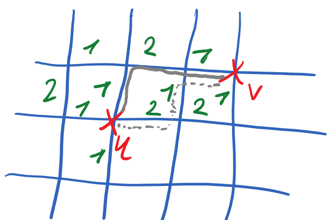
First passage perc. models a random metric space by assigning random "lengths" to the edges of $\mathbb{Z}^d$.

Let $(\eta_e)_{e \in \underbrace{E(\mathbb{Z}^d)}_{\text{edge set of } \mathbb{Z}^d}}$ be indep. and identically dist. with a common dist ν (weight dist.) supported on $[0,\infty)$.

For a given path P in the lattice, its length (or passage time) is the sum of η_e for the edges e in the path. Denote it by $T(P)$.

For two vertices $u, v \in \mathbb{Z}^d$, define the distance (or passage time) between them as

$$T(u,v) := \inf_{\substack{\text{paths } P \\ \text{between } u \text{ and } v}} T(P).$$



$T(u,v) = 4$.

We wish to study this metric space.

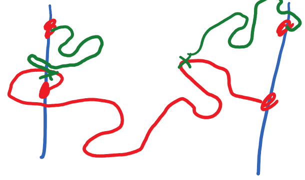
Questions: 1) Typical size of $T(u,v)$ for distant u, v?

2) Fluctuations around this typical size?

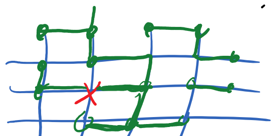
3) What is the geometry of the optimal path? (Called geodesic) How much does it deviate from a straight line?

4) How do different geodesics interact? Coalescence? geodesic tree?

Coalescence



5) How do the answers depend on the weight dist.? on the choice of lattice?



Geodesic tree = Tree of all geodesics from a given vertex

Universality?

Model introduced by Hammersley-Welsh (1965) and is very actively studied.

Auffinger-Damron-Hanson

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Reference: Auffinger-Damron-Hanson
50 years of first passage percolation.

Time constant

Convention: Define $T(x,y)$ for all $x,y \in \mathbb{R}^d$ by letting $T(x,y) = T(x',y')$ for the $x',y' \in \mathbb{Z}^d$ nearest to x,y (breaking ties arbitrarily).

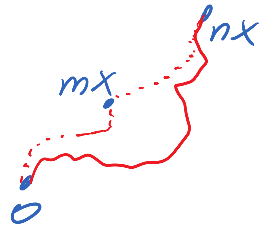
We want to study $T(0,x)$ for far away x.
Fix $x \in \mathbb{Q}^d$ (rational coord.) and study $T(0,n\cdot x)$ for large n.
We now show that $T(0,nx)$ grows linearly with n.

Thm.: For each non-zero $x \in \mathbb{Q}^d$ $\exists \mu(x)$ (the time constant) (deterministic)
s.t. $$\lim_{n\to\infty} \frac{T(0,nx)}{n} = \inf_n \frac{1}{n}\mathbb{E}(T(0,nx)) = \mu(x) \text{ a.s. and in } L^1$$

under the assumption that
$$\mathbb{E}\min\{t_1, \dots, t_{2d}\} < \infty \qquad (*)$$
where $t_1, \dots, t_{2d}$ are indep., with dist ν. (The weight dist.)

proof: We use the subadditive ergodic thm.
For convenience, suppose $x \in \mathbb{Z}^d \setminus \{0\}$.
Define a triangular array:
$$T_{m,n} := T(mx, nx), \quad 0 \le m < n.$$

Subadditivity: $T_{0,n} \le T_{0,m} + T_{m,n}$ a.s., for $0 \le m < n$.

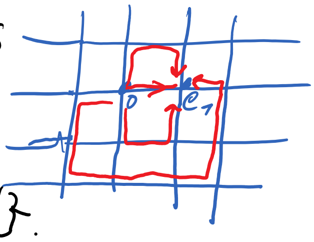


The invariance assump. of the theorem follow from the translation invariance in dist. of (η_e). Moreover, the stationary seq. are ergodic.

Also $T(0,nX) \ge 0$ so the lower bound assump. is verified.
Remains to check that $\mathbb{E}(T_{0,1}) < \infty$ ($= T(0,x)$)
Enough to prove this for x which is a neighbor of 0, by the subadditivity.
... is ... $\mathbb{E}t < \infty$ but $(*)$ also suffices

neighbor of 0, by the subadditivity.
Obvious if $\mathbb{E} t_e < \infty$. But $(*)$ also suffices.

Let $P_1, \dots, P_{2d}$ be $2d$ disjoint paths from 0 to x. Then



$$T(0,x) \le \min\{T(P_i) : 1 \le i \le 2d\}.$$

Let M be the biggest number of edges in one of the (P_i).

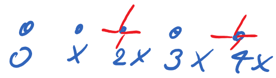
$$\mathbb{P}(T(P_i) > t) \le M\,\mathbb{P}\left(T(e_i) > \frac{t}{M}\right),$$ where e_i is some fixed edge

$$\Rightarrow \mathbb{P}(\min\{T(P_i) : 1 \le i \le 2d\} > t) =$$
$$= \prod_{i=1}^{2d} \mathbb{P}(T(P_i) > t) \le \left(M\,\mathbb{P}\left(T(e_1) > \frac{t}{M}\right)\right)^{2d}$$

By the assumption $\mathbb{E}\min\{T_{e_1}, \dots, T_{e_{2d}}\} < \infty$ and it remains to recall that for a non-neg. RV X, $\mathbb{E}X < \infty \iff \sum_{n=1}^{\infty} \mathbb{P}(X > n) < \infty$.

Remark: If $\mathbb{E}\min\{t_1, \dots, t_{2d}\} = \infty$ for IID t_i, $t_i \sim \nu$ then $\limsup_{n\to\infty} \frac{1}{n} T(0, nx) = \infty$ a.s.

We assume that $x \in \mathbb{Z}^d \setminus \{0\}$ for convenience.
Observe that the set of edges incident to $2nx$ is disjoint from the edges incident to $2mx$ for $n \ne m$.



Fix $\varsigma > 0$. Let $A_n := \{\min t_e \text{ for edges adjacent to } 2nx > \varsigma n\}$.

Then the events (A_n) are indep.
Moreover, $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \sum_{n=1}^{\infty} \mathbb{P}(\min\{t_1, \dots, t_{2d}\} > \varsigma n) = \infty$

$\mathbb{E}\min\{t_1, \dots, t_{2d}\} = \infty$

By the Borel-Cantelli lemma,
$$\mathbb{P}(\text{infinitely many } A_n \text{ occur}) = 1.$$
$$\Rightarrow \mathbb{P}\left(\frac{1}{2n} T(0, 2nx) > \frac{\varsigma}{2} \text{ for inf. many } n\right) = 1.$$

Since ς is arbitrary, $\limsup \frac{1}{n} T(0,nx) = \infty$ a.s.

$\Rightarrow \mathbb{P}\left(\frac{1}{2n} T(0, 2nx) > \frac{\zeta}{2} \text{ for } \ldots \text{ many } n\right) = 1.$

Since ζ is arbitrary, $\limsup_{n\to\infty} \frac{1}{n} T(0,nx) = \infty$ a.s.

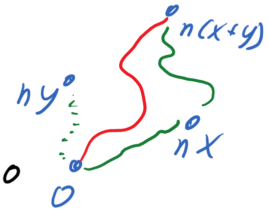
We now discuss further the time constant.

μ is apriori defined on $\mathbb{Q}^d$, and there it satisfies

μ is a norm:

1) $\mu(x+y) \le \mu(x) + \mu(y)$ (and $\mu \ge 0$)

2) $\mu(cx) = |c|\mu(x)$ for $c \in \mathbb{Q}$.



3) μ is Lipschitz continuous. I.e., $\exists \zeta > 0$ s.t. $|\mu(x) - \mu(y)| \le \zeta \|x-y\|_1$.

Thus μ can be extended to all of $\mathbb{R}^d$.

4) μ is sym. to symmetries of $\mathbb{Z}^d$ which preserve the origin.

the weight dist.

5) μ is non-zero everywhere if $\nu(\{0\}) < p_c(d)$ where $p_c(d)$ is the critical prob. for percolation on $\mathbb{Z}^d$.

Otherwise, μ is everywhere zero.

Remark: There is no explicit $\underbrace{\text{distribution}}_{\text{non-const.}}$ with $\nu(\{0\}) < p_c(d)$ for which μ is known.

The limit shape

First passage percolation gives a random metric space. How do balls look in this space?

$$B(t) := \{x \in \mathbb{Z}^d : T(0,x) \le t\}.$$

Theorem: Suppose the weight dist. ν satisfies

$$\mathbb{E} \min\{t_1, \ldots, t_{2d}\} < \infty \text{ for IID } t_i,\ t_i \sim \nu.$$

Then there exists $B_\nu \subseteq \mathbb{R}^d$ deterministic, convex, compact set s.t. for each $\varepsilon > 0$,

$$\mathbb{P}\left((1-\varepsilon)B_\nu \subseteq \frac{B(t)}{t} \subseteq (1+\varepsilon)B_\nu \text{ for all large } t\right) = 1.$$

Furthermore, B_ν has non-empty interior and is sym. around the axes.

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Goes back to Cox-Durrett (1981).

One can check simply that $B_\nu := \{x \in \mathbb{R}^d : \mu(x) \le 1\}$. the unit ball of the norm.

We will not show the proof of the theorem.

An important question: Is B_ν strictly convex? (in particular, does it have no flat edges? Note that if ν is supp. on a single value then μ is a multiple of the ℓ_1-norm and $B_\nu = \Diamond$)

Conjecture: If ν is an abs. cont. dist. then B_ν is strictly convex.

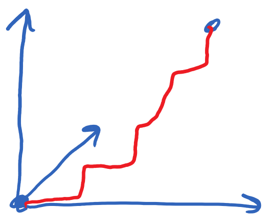
Question: Which shapes are possible as B_ν? (In particular, is there a ν for which B_ν is the ℓ_∞ unit ball $\square$?).

Variants of First Passage Percolation

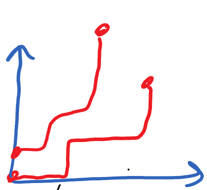
Directed (or oriented) **first passage percolation:**

paths may only move in the positive coordinate directions.

This simplifies the geometry of paths.



In this case, the number of edges on a path from 0 to x is exactly $\|x\|_1$. It allows to consider weight dist. which may be negative such as the Gaussian dist.



(it still makes sense to talk about the minimal passage time).

Directed last passage percolation: Maximize the sum of weights instead of minimizing.

Leads to integrable examples in dimension $d=2$! Geometric and exponential weights in directed last pass. perc. have been "solved"

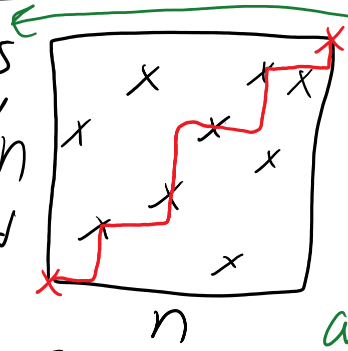
Geometric and exponential weights in directed last pass. perc. have been "solved" by combinatorial bijections and analytic techniques (breakthrough by Baik-Deift-Johansson (2000)).

Recently, a scaling limit "The directed landscape" was constructed by Dauvergne-Ortmann-Virág (2018).

Universality is expected, so the results learned in the solvable cases should hold in general.

Poissonian last passage percolation

Poisson process of unit intensity of points in the cube $[0,n]^d$



A random collection of points s.t. the number of points in a set A is dist. Poisson(Lebesgue meas. of A) and if $A \cap B = \emptyset$ then the numbers of points in A and in B are independent.

In two dimensions this is also "solvable".

Longest increasing subsequence problem:

Let π be a uniform random permutation in S_n.

Example: $n=7$ 4 2 1 5 7 6 3 $\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 5 & 7 & 1 & n \cdots & 2 \end{pmatrix}$

longest increasing subseq. has length 3.

Ulam's problem: Write T_n for the dist. of the longest increasing subseq. in π.

How big is $\mathbb{E}T_n$? Subadditive erg. thm. $\frac{T_n}{\sqrt{n}} \xrightarrow[n\to\infty]{} c_0$.

Vershik-Kerov, Logan-Shepp (1977?): $c_0 = 2$.

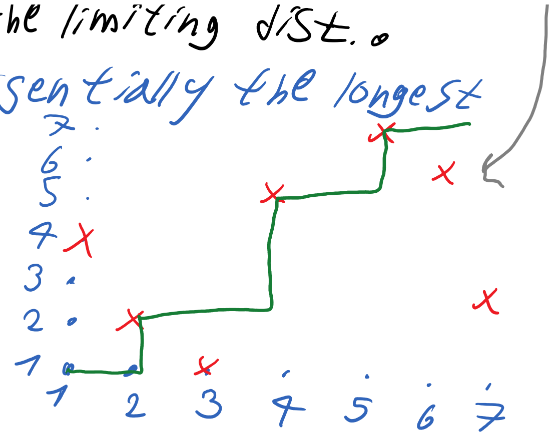
Baik-Deift-Johansson: Found the limiting dist.

Poisson LPP is essentially the longest

Baik–Deift–Johansson: Found the limiting dist.

In fact, Poisson LPP is essentially the longest increasing subseq. problem.

Draw the graph of π



In the other direction, one gets a permutation from a Poisson process in $[0,n]^2$ by sorting the x and y coord. of the points and thinking of the orig. coord. as a perm.

Reference: The surprising mathematics of longest increasing subsequences / Dan Romik.

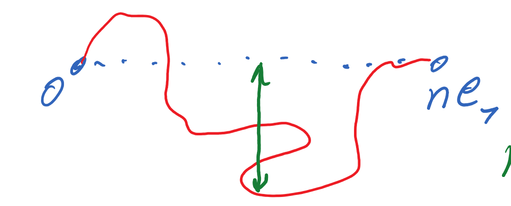
Predictions in two dimensions

First passage percolation: weight dist. ν is "nice".

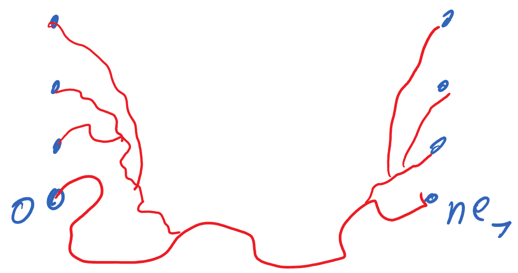
$$T_n := T(0, ne_1)$$

Conj.: $$\frac{T_n - n\mu(e_1)}{std(T_n)} \xrightarrow{d} F_2$$ Tracy-Widom dist.

$$Std(T_n) = c_0 \underbrace{n^{1/3}}_{n^{\chi}}(1+o(1)) \text{ as } n\to\infty$$



maximal deviation from straight line of order $n^{2/3} = n^{\xi}$



Geodesics started at vertical dist. $o(n^{2/3})$ will coalesce